ON THE EQUATIONS OF MOTION OF CONTROLLED MECHANICAL SYSTEMS

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In the present paper the study of controlled mechanical systems, which was started in [1],, is continued. The general questions in the analytic theory of controlled systems are considered. It is shown that the equations of motion of the controlled system may be written in all the fundamental forms: as Lagrange equations, as canonic equations, and as Appell's equations. The canonic transformations of holonomic controlled systems are considered. The equations of motion of nonholonomic controlled systems are derived.

The indices encountered in the paper take the following values:

$$\rho = 1,\ 2,\ \ldots,\ a;\quad \pi = 1,\ 2,\ \ldots,\ b;\quad i = 1,2,\ \ldots,\ 3n;\quad \alpha,\ \beta = 1,2,\ \ldots,\ s = 3n - a - b;$$
$$\tau,\ \sigma = 1,\ 2,\ \ldots,\ p;\quad \lambda, \mu = 1,\ 2,\ \ldots,\ l;\quad \vartheta = 1,\ 2,\ \ldots,\ p - l;\quad \gamma = 1,\ 2,\ \ldots, c;$$
$$\varkappa = 1,\ 2,\ \ldots,\ s-c;\quad \nu = 1,\ 2,\ \ldots,\ k$$

1. A system of n particles is moving relative to a fixed Cartesian coordinate system. We shall denote the coordinates and the mass of the first point of the system by x_1, x_2, x_3 and m_1, m_2, m_3, respectively, of the second point by x_4, x_5, x_6 and m_4, m_5, m_6 etc.

Let the system be subject to constraints among which there are some parametric ones. We shall denote the control parameters of the system by u_1, $u_2, \ldots, u_k$. Let

$$f_\rho (t,x_1, \ldots, x_{3n}) = 0, \qquad \varphi_\pi (t,x_1,\ldots,x_{3n}, u_1,\ldots, u_k) = 0 \qquad (1.1)$$

be the constraint equations of the system.

The constraints on the system will be taken as ideal. Then, the D'Alembert-Lagrange principle is valid for it [1]; for the true acceleration of the system, the relation

$$\sum (m_i x_i^{\cdot\cdot} - X_i)\ \delta x_i = 0 \qquad (1.2)$$

holds for all the possible displacements of the system. The latter are given [1] by the relations

$$\sum \frac{\partial f_\rho}{\partial x_i} \delta x_i = 0, \qquad \sum \frac{\partial \varphi_\pi}{\partial x_i} \delta x_i = 0 \tag{1.3}$$

The quantities $X_1, \ldots, X_{3n}$ in the relation (1.2) represent components along the coordinate axes of the active forces acting on the system. We shall take the quantities $X_1, \ldots, X_{3n}$ to be definite functions of time, of the system coordinates, and of the velocities of the system points, and also of the system control parameters. The latter assumption shows that the system admits of dynamic control as well as kinematic.

The advantage of controlled systems over uncontrolled ones is the possibility of action through the control parameters on the motion of the system. The control parameters, by their very nature, are undefined variables which may be dealt with arbitrary. By entering into the equations of motion of the controlled system (through the constraints on the system and through the forces acting on it), the control parameters essentially open up to the indicated system of equations. The equations of motion, therefore, by themselves do not determine the motion of the controlled system; this latter is determined only after there is given any conditions whatsoever which would permit us to close the system of equations of motion of the controlled system. Here is the source of the flexibility of the controlled systems.

2. Let $\chi_1, \ldots, \chi_s$ be some functions of time, of the coordinates, and perhaps, of the control parameters. We write Equations

$$\chi_\alpha(t,\ x_1,\ldots,x_{3n},\ u_1,\ldots,u_k) = q_\alpha \tag{2.1}$$

and use them to supplement the system of equations (1.1). Let us assume that the total number of Equations (1.1) and (2.1) equals $3n$, and that the functions χ_α are such that the system of equations (1.1) and (2.1) does not depend on the variables $x_1, \ldots, x_{3n}$. Then this system can be solved for $x_1, \ldots, x_{3n}$. Thus we have

$$x_i = x_i\ (t,\ q_1,\ldots,q_s,\ u_1,\ldots,u_k) \tag{2.2}$$

For fixed values of the control parameters there exists a one-to-one correspondence between the quantities $q_1, \ldots, q_s$ and the assumed constraints on the position of the system. By analogy with the mechanics of uncontrolled systems, we shall call the quantities $q_1, \ldots, q_s$ the generalized or Lagrangian coordinates of the controlled mechanical system being considered. Equalities (2.2) give explicit expressions for the Cartesian coordinate system in terms of its generalized coordinates, of time, and of the control parameters. It is not difficult to show that equalities (2.2) lead to the following explicit expressions for the possible displacements of the system

$$\delta x_i = \sum \frac{\partial x_i}{\partial q_\alpha} \delta q_\alpha \tag{2.3}$$

where $\delta q_1, \ldots, \delta q_s$ are arbitrary quantities.

Let us substitute Expressions (2.3) for the possible displacements of the system into the fundamental equation of mechanics (1.2). We get

$$\sum \delta q_\alpha \sum (m_i x_i'' - X_i) \frac{\partial x_i}{\partial q_\alpha} = 0 \tag{2.4}$$

Let us transform the left-hand side of this equation. First of all, from the equality

$$x_i' = \sum \frac{\partial x_i}{\partial q_\alpha} q_\alpha' + \frac{\partial x_i}{\partial t} + \sum \frac{\partial x_i}{\partial u_\nu} u_\nu'$$

obtained by differentiating (2.2) with respect to time, we find

$$\frac{\partial x_i}{\partial q_\alpha} = \frac{\partial x_i'}{\partial q_\alpha'}, \qquad \frac{d}{dt}\frac{\partial x_i}{\partial q_\alpha} = \frac{\partial x_i'}{\partial q_\alpha} \tag{2.5}$$

N o t e . It should be noted that here and in what follows, the control parameters, along with the system coordinates, are taken to be independent variables.

Taking equality (2.5) into account, we have

$$\sum m_i x_i'' \frac{\partial x_i}{\partial q_\alpha} = \frac{d}{dt} \sum m_i x_i' \frac{\partial x_i}{\partial q_\alpha} - \sum m_i x_i' \frac{d}{dt}\frac{\partial x_i}{\partial q_\alpha} =$$

$$= \frac{d}{dt} \sum m_i x_i' \frac{\partial x_i'}{\partial q_\alpha'} - \sum m_i x_i' \frac{\partial x_i'}{\partial q_\alpha} = \frac{d}{dt}\frac{\partial T}{\partial q_\alpha'} - \frac{\partial T}{\partial q_\alpha} \qquad \left(T = \frac{1}{2}\sum m_i x_i'^2\right)$$

Here T is the kinetic energy of the system. By introducing these expressions into relations (2.4), we write them as

$$\sum \delta q_\alpha \left(\frac{d}{dt}\frac{\partial T}{\partial q_\alpha'} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha\right) = 0 \qquad \left(Q_\alpha = \sum X_i \frac{\partial x_i}{\partial q_\alpha}\right) \tag{2.6}$$

Here the Q_α are the usual generalized forces. The quantities δq_α are arbitrary and, therefore, from (2.6) we find

$$\frac{d}{dt}\frac{\partial T}{\partial q_\alpha'} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \tag{2.7}$$

Thus, the equations of motion of the controlled system have the form of the usual second-order Lagrange equations.

If the forces acting on the system possess a force function U, then the generalized forces Q_α may be represented in the form $Q_\alpha = \partial U/\partial q_\alpha$, and Equations (2.7) reduced to the form

$$\frac{d}{dt}\frac{\partial L}{\partial q_\alpha'} - \frac{\partial L}{\partial q_\alpha} = 0 \tag{2.8}$$

where $L = T + U$ is the Lagrange function of the system.

The number of equations in each of the systems (2.7) and (2.8) equals the number of generalized coordinates. However, besides the generalized coordinates, these equations also contain the control parameters which still remain as completely undetermined variables. Therefore, the indicated systems of equations are open and, as noted at the end of the preceding Section, the motion of the system is not determined by them.

3. Let us denote

$$p_\alpha = \partial T / \partial q_\alpha' \tag{3.1}$$

and let us call the quantities p_α the generalized momenta of the system. Let us introduce into consideration the function

$$H^* = \sum p_\beta q_\beta' - T \tag{3.2}$$

With the aid of (3.1) we eliminate from it the generalized velocities q_β'. Then H^* will be a function of time, of the generalized coordinates and momenta, and also of the control parameters and their derivatives. Let us find the derivatives of H^* with respect to p_α and q_α. By differentiating H^* as a composite function and taking (3.1) into consideration, we get

$$\begin{aligned}\frac{\partial H^*}{\partial p_\alpha} &= \sum p_\beta \frac{\partial q_\beta'}{\partial p_\alpha} + q_\alpha' - \sum \frac{\partial T}{\partial q_\beta'}\frac{\partial q_\beta'}{\partial p_\alpha} = q_\alpha' \\ \frac{\partial H^*}{\partial q_\alpha} &= \sum p_\beta \frac{\partial q_\beta'}{\partial q_\alpha} - \frac{\partial T}{\partial q_\alpha} - \sum \frac{\partial T}{\partial q_\beta'}\frac{\partial q_\beta'}{\partial q_\alpha} = -\frac{\partial T}{\partial q_\alpha}\end{aligned} \tag{3.3}$$

Let us now take the Lagrange Equations (2.7). Using equality (3.1) it is written as

$$p_\alpha' = \frac{\partial T}{\partial q_\alpha} + Q_\alpha$$

By combining these equations with (3.3) we pass to the canonic equations

$$q_\alpha' = \frac{\partial H^*}{\partial p_\alpha}, \qquad p_\alpha' = -\frac{\partial H^*}{\partial q_\alpha} + Q_\alpha \tag{3.4}$$

Equations (3.4) have a form of the canonic Hamiltonian equations if the forces acting on the system admit of a force function. Indeed, by introducing in this case the Hamilton function $H = H^* - U$, we immediately get

$$q_\alpha' = \frac{\partial H}{\partial p_\alpha}, \qquad p_\alpha' = -\frac{\partial H}{\partial q_\alpha} \tag{3.5}$$

Thus, the equations of motion of the controlled system in the general case may be written in the form of canonic equations. However, if the forces acting on the system admit of a force function, then these equations reduce to the canonic Hamiltonian equations.

4. In the preceding Sections it was shown that the equations of motion of a controlled system may be written as second-order Lagrange equations and as canonic equations. Let us show that they may be written in a third fundamental form i.e. as Appell equations. To do this let us differentiate twice with respect to time, the expressions (2.2) for the Cartesian coordinates of the system points. We get the equalities

$$x_i'' = \sum \frac{\partial x_i}{\partial q_\alpha} q_\alpha'' + \ldots$$

where the dots stand for terms not depending on the second derivatives of the generalized coordinates. From these equalities we find

$$\frac{\partial x_i''}{\partial q_\alpha''} = \frac{\partial x_i}{\partial q_\alpha}$$

By substituting the latter identities into Equations (2.3) for the possible displacements of the system, we obtain

$$\delta x_i = \sum \frac{\partial x_i''}{\partial q_\alpha''} \delta q_\alpha$$

Using these equations let us transform the kinematic part of the fundamental equation of mechanics (1.2), We have

$$\sum m_i x_i'' \delta x_i = \sum m_i x_i'' \sum \frac{\partial x_i''}{\partial q_\alpha''} \delta q_\alpha = \sum \delta q_\alpha \sum m_i x_i'' \frac{\partial x_i''}{\partial q_\alpha''} = \sum \frac{\partial S}{\partial q_\alpha''} \delta q_\alpha$$

where by S we denote the acceleration energy of the system

$$S = \tfrac{1}{2} \sum m_i x_i''^2$$

On the other hand, on the basis of equalities (2.3), we find

$$\sum X_i \delta x_i = \sum X_i \sum \frac{\partial x_i}{\partial q_\alpha} \delta q_\alpha = \sum Q_\alpha \delta q_\alpha$$

where Q_α are the generalized forces.

The fundamental equation of mechanics can now be written as

$$\sum \left(\frac{\partial S}{\partial q_\alpha''} - Q_\alpha \right) \delta q_\alpha = 0 \tag{4.1}$$

Hence, because of the independence of the quantities δq_α we obtain the required equation

$$\frac{\partial S}{\partial q_\alpha''} - Q_\alpha = 0 \tag{4.2}$$

5. Let us now consider a canonic transformation of the equations of motion of the controlled mechanical system.

The problem is posed in the following way: among all the possible transformations

$$q_\alpha^* = q_\alpha^*(q, p, t, u); \quad p_\alpha^* = p_\alpha^* (q, p, t, u) \tag{5.1}$$

of the canonic variables q and p to $2n$ new variables q^* and p^*, to find such under which the Hamiltonian form of equations (3.5) of motion of the controlled mechanical system is preserved. The control parameters, which along with the generalized coordinates and momenta are variables of the system, participate in transformations (5.1) but are themselves not subject to transformation.

The peculiarity of the statement of the problem of the canonic transformations of the equations of motion of controlled mechanical systems, consists in the following: it is required to find such canonic transformations of a part of the variables of the system of equations of motion having a Hamiltonian form, that under these transformation the equations for the transformed variables preserve the Hamiltonian form.

The considered question is closely connected with Pfaffian forms; more precisely, to the properties of invariant connective forms and their adjoint system. Let us first consider this in a general formulation. Let us take any Pfaffian form

$$\omega = \Sigma A_\tau d\xi_\tau$$

where the A_τ are some functions of the variables ξ. The bilinear covariant of this form is given by the expression [2]

$$\Delta = \sum\left(\frac{\partial A_\tau}{\partial \xi_\sigma} - \frac{\partial A_\sigma}{\partial \xi_\tau}\right) d\xi_\tau \delta\xi_\sigma$$

where $d\xi$ and $\delta\xi$ are two groups of differentials of variables ξ. An important property of a bilinear covariant is its invariance to changes in the variables: the transformed bilinear covariant will be the bilinear covariant of the transformed form; in other words

$$\sum\left(\frac{\partial A_\tau^*}{\partial \xi_\sigma^*} - \frac{\partial A_\sigma^*}{\partial \xi_\tau^*}\right) d\xi_\tau^* \delta\xi_\sigma^* = \sum\left(\frac{\partial A_\tau}{\partial \xi_\sigma} - \frac{\partial A_\sigma}{\partial \xi_\tau}\right) d\xi_\tau \delta\xi_\sigma \qquad (5.2)$$

where

$$\xi_\tau^* = \xi_\tau^*(\xi_1, \ldots, \xi_p), \qquad \sum A_\tau^* d\xi_\tau^* = \sum A_\tau \, d\xi_\tau$$

The system of equations

$$\sum\left(\frac{\partial A_\tau}{\partial \xi_\sigma} - \frac{\partial A_\sigma}{\partial \xi_\tau}\right) d\xi_\tau = 0 \qquad (\sigma = 1, \ldots, p)$$

by which the bilinear covariant of the given Pfaffian form vanishes identically relative to the $\delta\xi_\sigma$, is called the adjoint system of the given form. As was the bilinear covariant, the adjoint system is invariantly (with respect to changes in the variables) connected with the form ω; the transformed adjoint system is the adjoint system of the transformed form.

The properties of the invariant connections of the bilinear covariant and the adjoint system with the Pfaffian form hold in conformity with the same transformation of the variables. Let us assume that the transformation affects only a part of the variables ξ. Let these be the first l variables. Then

$$\xi_\lambda^* = \xi_\lambda^*(\xi_1, \ldots, \xi_p), \qquad \xi^*_{l+\vartheta} = \xi_{l+\vartheta} \qquad (5.3)$$

Let us set $\delta\xi_{l+\vartheta} = d\xi_{l+\vartheta}$, which can be done because of the arbitrariness and independence of the differential $d\xi$ and $\delta\xi$. Let us show that in this case the bilinear covariant Δ vanishes by virtue of Equations

$$\sum\left(\frac{\partial A_\tau}{\partial \xi_\lambda} - \frac{\partial A_\lambda}{\partial \xi_\tau}\right) d\xi_\tau = 0 \qquad (\lambda = 1, \ldots, l) \qquad (5.4)$$

Indeed, it is not difficult to convince ourselves that because of the conditions $\delta\xi_{l+\vartheta} = d\xi_{l+\vartheta}$ and of Equations (5.4), the bilinear covariant reduces to

$$\Delta = \sum \left(\frac{\partial A_\lambda}{\partial \xi_{l+\vartheta}} - \frac{\partial A_{l+\vartheta}}{\partial \xi_\lambda} \right) d\xi_\lambda d\xi_{l+\vartheta}$$

On the other hand, from system (5.4) we find

$$\sum \left(\frac{\partial A_\lambda}{\partial \xi_{l+\vartheta}} - \frac{\partial A_{l+\vartheta}}{\partial \xi_\lambda} \right) d\xi_{l+\vartheta} = - \sum \left(\frac{\partial A_\lambda}{\partial \xi_\mu} - \frac{\partial A_\mu}{\partial \xi_\lambda} \right) d\xi_\mu$$

and, consequently, the bilinear covariant Δ is written as

$$\Delta = - \sum \left(\frac{\partial A_\lambda}{\partial \xi_\mu} - \frac{\partial A_\mu}{\partial \xi_\lambda} \right) d\xi_\mu d\xi_\lambda = 0$$

which is what was required. Let us now take the bilinear covariant

$$\Delta^* = \sum \left(\frac{\partial A_\tau^*}{\partial \xi_\sigma^*} - \frac{\partial A_\sigma^*}{\partial \xi_\tau^*} \right) d\xi_\tau^* \delta\xi_\sigma^*$$

of the transformed form ω. By what was just shown, this vanishes by virtue of Equations

$$\sum \left(\frac{\partial A_\tau^*}{\partial \xi_\lambda^*} - \frac{\partial A_\lambda^*}{\partial \xi_\tau^*} \right) d\xi_\tau^* = 0 \qquad (\lambda = 1, \ldots, l) \tag{5.5}$$

However, because of relations (5.2) the bilinear covariant Δ^* vanishes by virtue of Equations (5.4). Consequently, Equations (5.5) are equivalent to Equations (5.4).

The system of equations (5.4) is a subsystem of the adjoint system of form ω. By convention we shall call it the partial adjoint system of form ω in the variables $\xi_1, \ldots, \xi_l$. Then the obtained result can be formulated in the following way: under a partial transformation of variables (5.3) the partial adjoint system is invariantly connected with its own form.

Let us now return to the problem of the canonic transformations of the equations of motion of the controlled mechanical system.

Let us prove a theorem. If transformations (5.1) (in the space of the variables q, p, u, t) identically satisfy the relation

$$\sum p_\alpha^* \, dq_\alpha^* = \sum p_\alpha \, dq_\alpha + K dt + \sum K_\nu \, du_\nu + dW \tag{5.6}$$

where K, K_ν and W are some functions of the generalized coordinates of the momenta, of time, and of the control parameters, then transformations (5.1) are canonic. Indeed, identity (5.6), being rewritten as

$$\sum p_\alpha^* dq_\alpha^* - (H + K)\, dt - \sum K_\nu du_\nu - dW = \sum p_\alpha dq_\alpha - H dt$$

signifies that as a result of transformations (5.1) the form occurring on the right-hand side is transformed to the form occurring on the left-hand side.

It is easy to verify that the partial adjoint systems of these forms, under a transformation of variables, will be, respectively, the system (3.5) and the system

$$dq_\alpha^* - \frac{\partial (H+K)}{\partial p_\alpha^*} dt - \sum \frac{\partial K_\nu}{\partial p_\alpha^*} du_\nu = 0$$

$$dp_\alpha^* + \frac{\partial (H+K)}{\partial p_\alpha^*} dt + \sum \frac{\partial K_\nu}{\partial q_\alpha^*} du_\nu = 0 \qquad (5.7)$$

By virtue of the previously indicated property of invariant connection of the form and its partial adjoint system, the system (5.7) should be obtained as a result of transforming of the system (3.5).

On the other hand, system (5.7) has the Hamilton form

$$\frac{dq_\alpha^*}{dt} = \frac{\partial \Phi}{\partial p_\alpha^*}, \qquad \frac{dp_\alpha^*}{dt} = -\frac{\partial \Phi}{\partial q_\alpha^*} \qquad (\Phi = H + K + \sum K_\nu u_\nu') \qquad (5.8)$$

Hence transformations (5.1) are canonic, which is what was required.

The presence in (5.6) of the undetermined functions K and K_ν allows us to consider as a condition of canonic transformation, instead of (5.6), the simpler condition

$$\sum p_\alpha^* dq_\alpha^* = \sum p_\alpha dq_\alpha + dW \qquad (5.9)$$

in which, however, the differentiation is carried out for constant t and u_ν. Condition (5.9) is convenient for an explicit representation of the canonic transformations in terms of generating functions. Let us assume that transformations (5.1) are such that

$$\frac{\partial (q_1^*, q_2^*, \ldots, q_n^*)}{\partial (p_1, p_2, \ldots, p_n)} \neq 0$$

Then, the first group of equalities in (5.1) can be solved with respect to the canonic momenta p and, consequently, the variables q and q^* can be taken to be independent. Under these conditions, identity (5.9) gives

$$p_\alpha^* = \partial W / \partial q_\alpha^*, \qquad p_\alpha = -\partial W / \partial q_\alpha \qquad (5.10)$$

The function W is called the generating function of the canonic transformations. As a result of the canonic transformations, system (3.5) transforms to system (5.8).

Let us find expressions for the functions K and K_ν in terms of the generating function. For this we substitute Expressions (5.10) into identity (5.6). By expanding dW in it, we find

$$K = -\partial W / \partial t, \qquad K_\nu = -\partial W / \partial u_\nu$$

The new Hamilton function thus is

$$\Phi = H - \frac{\partial W}{\partial t} - \Sigma \frac{\partial W}{\partial u_\nu} u_\nu$$

Let us illustrate the results obtained by a simple example. Let there be given the canonic system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \qquad (H = qu + p)$$

Let us subject it to canonic transformations with a generating function $W = uqq^*$. Then the transformed system is

$$\frac{dq^*}{dt} = \frac{\partial \Phi}{\partial p^*}, \qquad \frac{\partial p^*}{dt} = -\frac{\partial \Phi}{\partial q^*} \qquad \left(\Phi = H - \frac{\partial W}{\partial u} u' = [qu + p - qq^* u']\right) \tag{5.11}$$

The square brackets here denote the passage to the new variables q^* and p^*. Let us show that this is indeed so. By substituting the expression for W into Equations (5.10) we find the explicit forms of the canonic transformations $p^* = qu$, $p = -q^* u$.

In the original system let us now expand the function H and in it pass to the new variables. We get

$$\frac{dq^*}{dt} = 1 - q^* \frac{u'}{u}, \qquad \frac{dp^*}{dt} = u + p^* \frac{u'}{u} \tag{5.12}$$

On the other hand, let us expand system (5.11). The function Φ in variables p^* and q^* is written as

$$\Phi = p^* - q^* u - q^* p^* \frac{u'}{u}$$

By substituting this into system (5.11) we again arrive at system (5.12). Which is what was required.

6. Up to this point the constraints on the system have been taken to be holonomic. Let us now assume that the constraint equations (including the parametric constraints) may depend on the velocities of the system points. Let $q_1, \ldots, q_s$ be the Lagrange coordinates of the system and let the system be subject to the nonholonomic constraints

$$F_\gamma (q_1, \ldots, q_s, q_1', \ldots, q_s', t, u_1, \ldots, u_k) = 0 \tag{6.1}$$

N o t e . Not all equations of system (6.1) need contain the control parameters. The form of (6.1) for the nonholonomic constraint equations of the system is taken solely in order to simplify the computations.

An example of a system with constraints of form (6.1) is the ordinary bicycle rolling without slippage on a horizontal plane. We can be convinced of this by setting up the constraint equations for the bicycle. The position of the bicycle, obviously, will be given if we are given the coordinates x and y of the center of the rear wheel of the bicycle, the angle ψ of the rotation of the bicycle frame around a vertical axis, the inclination θ of the bicycle frame to the horizon, the turn u of the steering bar (control parameter), and the angles φ and φ_1 of the turns around their own axes of, respectively, the rear and front wheels of the bicycle. For simplicity we shall consider that the bicycle remains vertical. In this case the constraint equations to which the rear wheel of the bicycle is subject, will be

$$x' + a\varphi' \cos\psi = 0, \qquad y' + a\varphi' \sin\psi = 0 \tag{6.2}$$

where a is the radius of the bicycle wheel. By taking into account that the coordinates of the center of the front wheel of the bicycle will be

$$x + b\cos\psi, \quad y + b\sin\psi$$

where b is the distance between the centers of the wheels, for the constraints on the front wheel of the bicycle we have the analogous equations

$$(x + b\cos\psi)' + a\varphi_1' \cos(\psi + u) = 0, \qquad (y + b\sin\psi)' + a\varphi_1' \sin(\psi + u) = 0$$

These equations are easily transformed to the following final form

$$\varphi' + \varphi_1' \cos u = 0, \qquad b\psi' + a\varphi_1' \sin u = 0 \tag{6.3}$$

Equations (5.2) and (6.3) form a complete system of nonholonomic constraints for the bicycle. Obviously, they have the form of Equations (6.1).

Summarizing the definition of possible displacement as taken for holonomic controlled systems (relations (1.3)), let us define the possible displacements of controlled systems in the presence of constraints (6.1), by the relations

$$\sum \frac{\partial F_\gamma}{\partial q_\alpha'} \delta q_\alpha = 0 \tag{6.4}$$

Let us find the equations of motion of the system under constraints (6.1). After introduction of Lagrange coordinates, the fundamental equation of mechanics is written as

$$\sum \left(\frac{d}{dt} \frac{\partial T}{\partial q_\alpha'} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha \right) \delta q_\alpha = 0 \tag{6.5}$$

In case the constraints (6.1) are absent, all the δq_α are independent, and from relation (6.5) follows the Lagrange equations for holonomic systems. In the considered case of a nonholonomic system, the quantities δq_α are no longer independent. They are constrained by relations (6.4). In this case the equations of motion of the system with multipliers are easily obtained from (6.5). They have the form

$$\frac{d}{dt} \frac{\partial T}{\partial q_\alpha'} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha + \sum \lambda_\gamma \frac{\partial F_\gamma}{\partial q_\alpha} \tag{6.6}$$

where λ_γ are the multipliers, subject to determination.

The equations of motion of nonholonomic controlled systems may be written also in the form of Appell equations. To this end, let us supplement Equations (6.1) by the relations

$$\Phi_\varkappa (q_1, \ldots, q_s, q_1', \ldots, q_s', t, u_1, \ldots, u_k) = \omega_\varkappa \tag{6.7}$$

in such a way that the system of relations (6.1) and (6.7) will be solvable with respect to the derivatives $q_1', \ldots, q_s'$. By taking into account that the quantities $\omega_\varkappa$ denote the numerical values of the functions $\Phi_\varkappa$ for values of the arguments satisfying the constraint equations (6.1), we conclude by virtue of the assumptions made with respect to (6.7) that relations (6.1) and (6.7) establish a one-to-one correspondence between the arbitrary $\omega_\varkappa$ and the manifold of kinematically admissible velocities of the system. On the other hand, this correspondence is given by (6.7), on the other, by the relations

$$q_\alpha' = \varphi_\alpha (q_1, \ldots, q_s, \omega_1, \ldots, \omega_{s-c}, t, u_1, \ldots, u_k) \tag{6.8}$$

which are obtained by solving relations (6.1) and (6.7).

Let us define the quantities $\delta\vartheta_\varkappa$ by the equalities

$$\delta\vartheta_\varkappa = \sum \frac{\partial \Phi_\varkappa}{\partial q_\alpha'} \delta q_\alpha \tag{6.9}$$

Relations (6.9), together with (6.4), establish a one-to-one corresponence between the arbitrary quantities $\delta\vartheta_\varkappa$ and the possible displacements of the system. Indeed, a specific system of quantities $\delta\vartheta_\varkappa$. by virtue of (6.9) corresponds to every possible displacement of the system. On the other hand, the determinants of systems (6.4) and (6.9) will be the Jacobians of systems (6.1) and (6.7), and these latter are different from zero. Consequently, the system of relations (6.4) and (6.9) may be solved with respect to δq_α and hence to each system of values $\delta\vartheta_\varkappa$ there corresponds a possible displacement of the mechanical system.

It is not difficult to see that the equalities expressing δq_α in terms of $\delta\vartheta_\varkappa$, may be written as

$$\delta q_\alpha = \sum \frac{\partial \varphi_\alpha}{\partial \omega_\varkappa} \delta\vartheta_\varkappa \tag{6.10}$$

To prove this let us substitute the latter expression for δq_α into relation (6.4), we get

$$\sum \frac{\partial F_\gamma}{\partial q_\alpha'} \frac{\partial \varphi_\alpha}{\partial \omega_\varkappa} \delta\vartheta_\varkappa = 0 \tag{6.11}$$

But by (6.4) and (6.8)

$$\sum \frac{\partial F_\gamma}{\partial q_\alpha'} \frac{\partial \varphi_\alpha}{\partial \omega_\varkappa} = 0$$

Therefore, equality (6.11) is satisfied identically with respect to $\delta\vartheta_\varkappa$. Which is what we required.

Let us introduce the acceleration energy $S\ (q_1, q', q'', t, u, u')$. Then the fundamental equation of mechanics may be written in the form (4 1).

Let us note that from equality (6.8) follows the identity

$$\frac{\partial \varphi_\alpha'}{\partial \omega_\varkappa'} = \frac{\partial \varphi_\alpha}{\partial \omega_\varkappa}$$

and, consequently, equality (6.10) may be written as

$$\delta q_\alpha = \sum \frac{\partial \varphi_\alpha'}{\partial \omega_\varkappa'} \delta\vartheta_\varkappa \tag{6.12}$$

In the assumptions of equalities (6.10) and (6.12) let us now transform the fundamental equation of mechanics (4.1). We have

$$\sum \left(\frac{\partial S}{\partial q_\alpha''} - Q_\alpha \right) \delta q_\alpha = \sum \frac{\partial S}{\partial q_\alpha''} \frac{\partial \varphi_\alpha'}{\partial \omega_\varkappa'} \delta\vartheta_n - \sum Q_\alpha \frac{\partial \varphi_\alpha}{\partial \omega_\varkappa} \delta\vartheta_\varkappa =$$

$$= \sum \left(\frac{\partial S^*}{\partial \omega_\varkappa'} - Q_\varkappa^* \right) \delta\vartheta_\varkappa = 0 \quad \left(Q_\varkappa^* = \sum Q_\alpha \frac{\partial \varphi_\alpha}{\partial \omega_\varkappa} \right) \tag{6.13}$$

where S^* is the acceleration energy transformed to the variables $\omega_\varkappa$.

Equality (6.13) is satisfied for all $\delta\vartheta_\varkappa$. By taking into account that the latter are completely arbitrary, from (6.13) we get

$$\frac{\partial S^*}{\partial \omega_\varkappa'} - Q_\varkappa^* = 0$$

This is the desired equation. The choice of quantities $\omega_\varkappa$ is connected with relations (6.7), which practically are given arbitrarily. Therefore, the independents of the derivatives q_α' may be taken as the quantities $\omega_\varkappa$. In this case the equations of motion of the controlled system under consideration can be written completely in terms of a Lagrange coordinate system.

BIBLIOGRAPHY

1. Kirgetov, V.I., O kinematicheski upravliaemykh mekhanicheskikh sistemakh (On kinematically controlled mechanical systems). *PMM* Vol.28,№1,1964.
2. Levi-Civita, T. and Amaldi, U., Kurs teoreticheskoi mekhaniki (Course in theoretical mechanics). IL, Vol.2, Pt.2, 1951.

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